Lecture 4

• Outline:
  • Nonlinear dynamics in the phase plane
    - Phase portraits
    - Fixed points and linearizations
    - Conservative systems
    - Reversible systems
    - Index theory
Phase Portraits (1)

- 2d vector fields:
  \[
  \begin{align*}
  \dot{x}_1 &= f_1(x_1, x_2) \\
  \dot{x}_2 &= f_2(x_1, x_2)
  \end{align*}
  \]
  \[\dot{\vec{x}} = \vec{f}(\vec{x})\]

- Entire phase plane filled with trajectories

- Qualitative behaviour in phase plane?
Phase Portraits (2)

- Salient features:
  - Fixed points (A, B, C)
  - Closed orbits (D)
  - Arrangement of trajectories near fixed points
  - Stability of fixed points and closed orbits – do traj. converge on them or move away?
How to construct a phase portrait (1)

- Let's consider a simple example:
  \[
  \begin{align*}
  \dot{x} &= x + \exp(-y) \\
  \dot{y} &= -y
  \end{align*}
  \]

- Qualitative arguments via nullclines: \( \dot{x} = 0, \; \dot{y} = 0 \)
How to construct a phase portrait (2)

- Numerically – Runge Kutta (replace 1d variables in Lecture 2 by the respective vectors)
- Direction field + plot selected trajectories
Existence and Uniqueness of Solutions and Implications

- **THEOREM:** Consider $\frac{dx}{dt} = f(x)$, $x(0) = x_0$ and suppose that $f$ and all its partial derivatives are continuous in some open set around $x(0)$. Then some time interval $(-\tau, \tau)$ exists in which the initial value problem has a unique solution.

- **Consequences:**

  - P-B-Theorem
  - Trajectories don't intersect
Fixed Points and Linearizations (1)

- Suppose that
  \[ \dot{x} = f(x, y) \quad \dot{y} = g(x, y) \]  
  (1)

- Has a FP at \((x^*, y^*)\), i.e.
  \[ 0 = f(x^*, y^*) \]  
  \[ 0 = g(x^*, y^*) \]  
  (2)

- What is the fate of small perturbations \(\epsilon\) and \(\eta\) from the fixed points? \(\epsilon = x - x^*\) \(\eta = y - y^*\)  
  (3)

- (3) into (1), first order Taylor expansion, use (2):
  \[ \dot{\epsilon} = \dot{x} = f(x^* + \epsilon, y^* + \eta) \]
  \[ \dot{\epsilon} \approx f(x^*, y^*) + \epsilon \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + O(\epsilon^2, \eta^2, \epsilon \eta) \]
Fixed Points and Linearizations (2)

- Similarly:
  \[ \dot{\eta} = \dot{y} = g(x^* + \epsilon, y^* + \eta) \]
  \[ \dot{\eta} \approx g(x^*, y^*) + \epsilon \frac{\partial g}{\partial x} x + \eta \frac{\partial g}{\partial y} y + O(\epsilon^2, \eta^2, \epsilon \eta) \]

- We obtain 2 linear diff. Eq's in \( \epsilon \) and \( \eta \):
  \[
  \begin{pmatrix}
  \dot{\epsilon} \\
  \dot{\eta}
  \end{pmatrix} = 
  \begin{pmatrix}
  \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\
  \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}
  \end{pmatrix}
  \begin{pmatrix}
  \epsilon \\
  \eta
  \end{pmatrix}
  \]
  \[
  \begin{pmatrix}
  \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\
  \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}
  \end{pmatrix}_{(x^*,y^*)}
  \begin{pmatrix}
  \epsilon \\
  \eta
  \end{pmatrix}
  \]

  Jacobian matrix

- Can now proceed as in Lecture 3
Fixed Points and Linearizations (3)

• What about nonlinear corrections $O(\varepsilon^2, \eta^2, \varepsilon \eta)$?
  • General rule: linearization is fine as long as fixed point for linearized system is no borderline case (center, degenerate node, star, non-isolated FP)

• Let's have a look at an example to see how it works!

\[
\begin{align*}
\dot{x} &= -y + ax(x^2 + y^2) \\
\dot{y} &= x + ay(x^2 + y^2)
\end{align*}
\]

all $a$: $(x^*, y^*) = (0,0)$

• FPs? $a < 0$: $(x^*, y^*) = (\pm (-a)^{-3/4} (1-a)^{1/2}, \pm (-a)^{1/4} (1-a)^{1/2})$
Fixed Points and Linearizations (4)

- Jacobian at (0,0)?
  \[
  \begin{pmatrix}
  \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\
  \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}
  \end{pmatrix}_{(0,0)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
  \]

- Eigenvalues?
  - Characteristic Eq.? \( \lambda^2 + 1 = 0 \) \( \lambda = \pm i \)
  - This predicts nonlinear centers at (0,0) independent of a.
  - Centers are a “borderline case” -> have to be cautious
  - Higher order corrections?
Fixed Points and Linearizations (5)

- What to do about the fully nonlinear system?
  - Generally difficult
  - In this case: polar coordinates!
    \[ x = r \cos \Theta \quad y = r \sin \Theta \]

- ... some algebra ...
  \[ \dot{\Theta} = 1 \quad \dot{r} = ar^3 \]

- This system is far easier to analyze
  - \(a<0\) -> stable spiral, \(a>0\) -> unstable spiral
  - \(a=0\) nonlinear centers
  - Result is not independent of \(a\) as predicted by Lin.!
Hyperbolic Fixed Points

• For an n-dimensional system a fixed point is called **hyperbolic**, if for all eigenvalues \( i=1,\ldots,n \) it holds \( \text{Re}(\lambda_i)\neq 0 \)

• Hartman-Grobman Theorem:
  • Local phase portrait near a hyperbolic fixed point is “topologically equivalent” to the phase portrait of its linearization
  • “topologically equivalent” \(~\text{homeomorphic mapping exists}\)

• Structural stability: phase portrait is ss if its topology cannot be changed by small perturbation to vector field
  • Saddle point? -> ss, center? -> not ss
Linearizations

- When are linearizations for marginal cases OK?
  - In general, difficult to say and needs to be checked in each case
- For special classes of systems more general results that can help are available:
  - Conservative systems
  - Reversible systems
Conservative Systems

- Consider a particle of mass $m$ moving along the $x$-axis
  - Newton -> $m \ddot{x} = F(x)$
  - Potential energy: $-dV/dx = F(x)$

$$m \ddot{x} + dV/dx = 0$$
$$m \dddot{x} + dV/dx \dot{x} = 0$$
$$\frac{d}{dt}(m \dot{x}^2 + V(x)) = 0 \quad E = m \dot{x}^2 + V(x) \rightarrow dE/dt = 0$$

- DEF: A **conserved quantity** $E$ is a real-valued function that is constant on trajectories $dE/dt=0$ (and nonconstant on every open set)
- Conservative systems? A cons. quantity exists.
Conservative Systems (2)

• Conservative systems cannot have any attracting fixed points!
  • Suppose $x^*$ is FP -> all points in basin have same energy -> energy is constant in basin -> contradiction.

• Let's have a look at an example, a particle with mass $m=1$ in a potential $V(x) = -1/2 x^2 + 1/4 x^4$
  \[
  \begin{align*}
  \dot{x} &= y \\
  \dot{y} &= x - x^3
  \end{align*}
  \]

• FPs? $x=0, \pm 1; y=0$
Conservative Systems (3)

- Stability? -> Jacobian

\[
\begin{pmatrix}
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\
\frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}
\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1-3x^2 & 0 \end{pmatrix}
\]

- (0,0): \( \lambda^2 - 1 = 0 \) \( \lambda = \pm 1 \) \( \rightarrow \) saddle node

- (±1,0): \( \lambda^2 + 2 = 0 \) \( \lambda = \pm \sqrt{2}i \) \( \rightarrow \) centers
  - Warning! But conserved quantity!
  - Trajectories are contours of E=const.

- How does phase portrait look like?
Conservative Systems (4)

Homoclinic orbits: trajectories that start and end at the same FP

Heteroclinic orbits: trajectories that connect FPs
Conservative Systems (5)

- THEOREM (nonlinear centers in cons. Syst.)
  - Consider $\frac{dx}{dt}=f(x)$ and $f$ cont. differentiable and suppose a conservative quantity $E(x)$ exists and $x^*$ is an isolated FP. If $x^*$ is a local minimum of $E$ then all trajectories that are sufficiently close to $x^*$ are closed.
  - “Proof”:
    - Each trajectory is contained in some contour of $E$
    - Near local optima these contours are closed
    - Does trajectory go all round contour or stop at a FP?
      - No, since $x^*$ is an isolated FP.
Reversible Systems

- Time reversal symmetry: “can't distinguish whether system runs forward or backward”
- Any mechanical system \( m \ddot{x} = F(x) \) is invariant under \( t \rightarrow -t \)
- Write this in our usual form: \( \dot{x} = y \quad \dot{y} = 1/m F(x) \)
- Invariance to \( t \rightarrow -t \) then implies:
  
  if \( (x(t), y(t)) \) is solution so is \( (x(-t), -y(-t)) \)

- DEF:
  
  A second order system is **reversible** if it is invariant to \( t \rightarrow -t \) and \( y \rightarrow -y \).
THEOREM (Nonlinear centers for rev. systems)

Suppose \((x^*, y^*) = (0, 0)\) is a linear center for a cont. diff. 2\(^{nd}\) order system and suppose the system is reversible. Then sufficiently close to \((0, 0)\) all trajectories are closed curves.

“Proof”:

Consider trajectory sufficiently close to origin
Index Theory

- So far ... linearizations -> “local methods”
- Index theory: global information about the phase portrait
  - E.g.: Must a closed trajectory always encircle a fixed point? What types of FP are permitted?
- Index of a closed curve:
  - C “simple” closed curve, no FP on C
  - $\phi = \tan^{-1}(dx/dt/dy/dt)$
  - X moves counterclockwise around C, measure change in $\phi$
  - Must be integer multiple of $2\pi$
    $$I_C = \frac{1}{2} \pi \left[ \phi \right]_C$$
Index of a Closed Curve

\[ I_c = +1 \]
Examples (1)

$|c| = -1$
Examples (2)

Find the index of the vector field \((\dot{x}, \dot{y}) = (yx^2, x^2 - y^2)\) on the curve \(1 = x^2 + y^2\)

\(I_c = -\pi + 2\pi - \pi = 0\)
Properties of Indices

- If $C$ is deformed continuously into $C'$ without touching a FP then $I_C = I_{C'}$.
- If $C$ doesn't enclose a FP then $I_C = 0$
  - deform curve to a point
- If all arrows in the VF are reversed by $t \rightarrow -t$ then the index stays the same
  - $\phi \rightarrow \phi + \pi$
- If $C$ is a trajectory for the system, then $I_C = +1$.
  - VF is everywhere tangent to curve.
Indices of Points

- Can deform curves around a point to a point and thus define indices of points

stable node  unstable node  saddle node

\[ I_c = +1 \]  \[ I_c = +1 \]  \[ I_c = -1 \]
Index Theory

- **THEOREM**: If a closed curve surrounds $n$ isolated fixed points $1 \ldots n$ then

$$ I_C = |I_1| + |I_2| + |I_3| + \ldots + |I_n| $$
Index Theory

• THEOREM: Any closed orbit in the phase plane must enclose fixed points whose indices sum up to +1.

• Remarks:
  • There is always at least one FP inside any closed orbit in the phase plane!
  • If there is only one FP inside, it cannot be a saddle node!
Index Theory

• Show that \( (\dot{x}, \dot{y}) = (xe^{-x}, 1+x+y^2) \) has no closed orbit!

• Has no FP \( \rightarrow \) cannot have a closed orbit by last theorem.
Summary

- Linearizations, fixed points
- Conservative systems
- Reversible systems
- Index Theory