• Outline:
  • Limit Cycles
    - Definition and examples
    - How to rule out limit cycles
      • Gradient systems
      • Liapunov functions
      • Dulacs criterion
    - Poincare-Bendixson theorem
    - Hopf bifurcations
    - Poincare maps
Limit Cycles

- **DEF:** A **limit cycle** is an *isolated* closed trajectory.

- **Examples:**
  - Heart beat, pacemaker neurons, daily rhythms in body temperature, chemical reactions, ...
Limit Cycles

- Are inherently a non-linear phenomenon
  - What about closed orbits in linear systems?
    \[
    \dot{x} = A \ddot{x} \quad \text{purely complex eigenvalues}
    \]
  - They can have solutions that are closed orbits!
  - Linearity implies: if \( x(t) \) is a solution so is \( c \cdot x(t) \)
  - Amplitude is determined by ICs, any small perturbation persists forever
  - In contrast: in limit cycles in nonlinear systems the structure of the system determines the amplitude and shape of the limit cycle.
Examples (1)

- Not hard to construct examples of limit cycles in polar coordinates

\[
\dot{\theta} = 1, \\
\dot{r} = r(1-r^2)
\]

\(x(t), y(t)\) -> sine or cosine, 
- periodic functions in \(x\) 
And \(y\) directions
Van der Pol Oscillator

\[ \ddot{x} + \mu (x^2 - 1) \dot{x} + x = 0 \]

nonlinear damping term

- Equation arose in connection with nonlinear electrical circuits used in the first radios

- Damping term
  - Causes large amplitude oscillations to decay, but pumps them back if they become too small

- VdP osci has a unique stable LC for each \( \mu > 0 \)
What to do?

• Suppose we want to analyze a given 2d flow
  • Rule out existence of limit cycles
    - Index theory (last lecture)
    - We know the system belongs to a class of systems that cannot have LCs (→ e.g. Gradient systems)
    - Construct Liapunov function
    - Dulac's criterion
    - ...
  • Prove existence of a LC
    - Poincare-Bendixson theorem
DEF: A gradient system is a system that can be written in the form $\dot{x} = -\nabla V$. $V$ is then called a potential function.

Remarks:

- In components:
  \[
  \dot{x} = -\partial_x V(x,y) \\
  \dot{y} = -\partial_y V(x,y)
  \]

- Every 1d system is a gradient system

- How to check if a 2d system is a gradient system?
  If $\dot{x} = f(x,y)$ is a gs, then: $\partial f / \partial y = \partial g / \partial x$
  $\dot{y} = g(x,y)$
Example Gradient Systems

\[
\begin{align*}
\dot{x} &= y + 2xy \\
\dot{y} &= x + x^2 - y^2
\end{align*}
\]  \hspace{1cm} (1)

- Is (1) a gradient system? If so, find a potential.
- Check condition \( \frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} \)

\[
\begin{align*}
\frac{\partial f}{\partial y} &= 1 + 2x \\
\frac{\partial g}{\partial x} &= 1 + 2x
\end{align*}
\]

\( \checkmark \)

- Find potential? Partial integration.

\[
\begin{align*}
-V_x &= y + 2xy \\
-V_y &= x + x^2 - C_y(y) = \dot{y} = x + x^2 - y^2
\end{align*}
\]

\[
\begin{align*}
dC/dy &= y^2 \\
C(y) &= 1/3 y^3
\end{align*}
\]

\( \rightarrow \)

\[ V(x, y) = -xy - x^2 y + 1/3 y^3 \]
Gradient Systems

THEOREM: Closed orbits are impossible in gradient systems.

Proof:

Suppose there is a closed orbit. Consider change in $V$ after one circuit.

- On the one hand: $\Delta V=0$ ($V$ is a scalar)
- On the other hand:

$$
\Delta V = \int_0^T \frac{dV}{dt} \, dt = \int_0^T (\nabla V \circ \dot{x}) \, dt
$$

$$
\Delta V = -\int_0^T \|\dot{x}\|^2 \, dt < 0 \quad \text{(contradiction)}
$$
Liapunov Functions

- Occasionally possible to find “energy-like” functions that decrease along trajectories.
- **DEF**: Consider a system $dx/dt=f(x)$ with a fixed point at $x^*$. A Liapunov function is a cont. diff. real-valued function $V(x)$, such that:
  - $V(x)>0$ for all $x \neq x^*$ and $V(x^*)=0$ ($V$ is pos. definite)
  - $dV/dt<0$ for all $x \neq x^*$ (all trajectories are “downhill towards $x^*$)

Then $x^*$ is globally asymptotically stable.

- In particular the system has no limit cycles.
Liapunov Functions

• “Proof”: Why does the existence of a Liapunov function rule out closed orbits?
  • Consider change of $V$ along a closed orbit.
  • $V$ is real-valued, after one circuit one should have $\Delta V=0$
  • On the other hand:
    \[
    \Delta V = \int_0^T \frac{dV}{dt} \, dt < 0
    \]

• Existence of a LF implies that trajectories move monotonically down the graph of $V$ toward $x^*$
Problem: How to find Liapunov functions?
No general method, requires divine inspiration :-)
Often sums of squares work ...

\[
\begin{align*}
\dot{x} &= -x + 4y \\
\dot{y} &= -x - y^3
\end{align*}
\]  \hspace{1cm} (2)

Ansatz: \[ V(x, y) = x^2 + ay^2 \]

\[
\dot{V} = 2x\dot{x} + 2ay\dot{y} = 2x(-x + 4y) + 2ay(-x - y^3) \\
= -2x^2 + 8xy - 2ay^4 - 2axy
\]

Choose \( a = 4 \)

\[ = -2x^2 - 8y^4 \]
Liapunov Functions

- Is $V(x, y) = x^2 + 4y^2$ a Liapunov function for (2)?
  - $V > 0$ for all $(x, y) \neq (x^*, y^*) = (0, 0)$
  - $V(0, 0) = 0$
  - $dV/dt < 0$ for $(x, y) \neq (0, 0)$ by construction.
- Theorem above then implies that $(0, 0)$ is globally asymptotically stable.
Dulac's Criterion

- Let $\frac{dx}{dt} = f(x)$ be a cont. diff. VF in a simply connected subset $R$ of the plane. If there exists a cont. diff. real-valued function $g(x)$ such that $\nabla \circ (g \dot{x})$ has one sign throughout $R$ then there are no closed orbits lying entirely in $R$.

$\int\int_A \nabla \circ (g \dot{x}) dA = \oint_C g \dot{x} \circ \vec{n} dl$

$\neq 0$

$= 0$ since $\dot{x} \perp \vec{n}$
Dulac in Practice ...

- Requires a real-valued function $g$ with the above properties
- Same problem as with Liapunov functions
  - No general method to find $g$, divine inspiration ...

- So far: negative results (ruling out limit cycles). How to prove that limit cycles must exist?
Poincare Bendixson Theorem

• Suppose
  • R is a closed bounded subset of the plane and $dx/dt = f(x)$ is a cont. diff. VF on an open set containing R
  • R does not contain any fixed points
  • There exists a traj. that is “confined” in R
• Then, either C is a closed orbit or C spirals toward one. R contains a closed orbit.
P-B-T

- Limits types of behaviour in the phase plane:
  - Fixed points, oscillations
  - Self-sustaining oscillations, but no chaos!

- PBT in applications:
  - First two conditions are easy to show
  - Often hard to construct a “trapping region” to prove existence of a confined trajectory
  - Situation often simpler, if system has simple representation in polar coordinates
P-B-T in Action

- Consider:  
  \[ \dot{r} = r(1-r^2) + \mu r \cos \Theta \]
  \[ \dot{\Theta} = 1 \]

- Does the closed orbit at \( r=1 \) survive for small \( \mu \)?

\[ r_{\text{max}} (1-r_{\text{max}}^2) + \mu r_{\text{max}} \cos \Theta < 0 \]
  e.g. \( r_{\text{max}} > \sqrt{1+\mu} \)

\[ r_{\text{min}} (1-r_{\text{min}}^2) + \mu r_{\text{min}} \cos \Theta > 0 \]
  e.g. \( r_{\text{min}} < \sqrt{1-\mu} \)

There must be a limit cycle in
\[ 0.999 \sqrt{1-\mu} < r < 1.001 \sqrt{1+\mu} \]
Hopf Bifurcations

• How can a FP lose stability?

Stable node
Transcritical, Saddle node, Pitchfork bifurcations

Stable spiral
Hopf bifurcations
Supercritical Hopf Bifurcation

- Below bif. point: small oscillations gradually abating

- Above bif. point: small oscillations gradually built up to small amplitude

- Stable spiral changes into an unstable spiral surrounded by a small elliptical limit cycle
Supercritical Hopf – Example (0)

\[
\begin{align*}
\dot{r} &= \mu r - r^3 \\
\dot{\Theta} &= \omega + br^2
\end{align*}
\]

- $\mu < 0$: origin is stable spiral
- $\mu = 0$: origin still is “weak” stable spiral
- $\mu > 0$: origin unstable spiral + limit cycle at $r = \sqrt{\mu}$

$\mu$ controls stability of FP
$\omega$ controls frequency of rotations
$B$ controls dependence of frequency on amplitude for large amplitudes
Supercritical Hopf – Example (1)

• What happens to eigenvalues?
  • In cartesian coordinates ...

\[ \begin{align*}
  x &= r \cos \Theta \\
  y &= r \sin \Theta \\
  \dot{x} &= r \cos \Theta - r \dot{\Theta} \sin \Theta \\
  &= (\mu r - r^3) \cos \Theta - r (\omega + br^2) \sin \Theta \\
  &= (\mu - [x^2 + y^2]) x - (\omega + b[x^2 + y^2]) y \\
  &= \mu x - \omega y + \text{cubic terms} \\
  \dot{y} &= \omega x + \mu y + \text{cubic terms} \\
  J &= \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix} \\
  \lambda &= \mu \pm i\omega
\end{align*} \]
Example illustrates two general rules:

- Size of LC grows continuously from zero $\propto \sqrt{\mu - \mu_c}$
- Frequency of limit cycle at birth is given by $\text{Im}(\lambda)$ at $\mu_c$

However:

- Eigenvalues generally don't cross $x=0$ in straight lines
- At birth limit cycle generally is an ellipse (not circle as in example)
Subcritical Hopf Bifurcations

\[ \dot{r} = \mu r + r^3 - r^5 \quad \text{Cubic term now destabilizing!} \]

\[ \dot{\Theta} = \omega + br^2 \]
Poincare maps

- Useful for studying swirling flows
- $\frac{dx}{dt} = f(x)$ $n$ dim. flow, $S$ $n-1$ dim. surface of section transverse to flow
- Poincare map is a mapping of $S$ to itself obtained by following trajectories from one intersection with $S$ to the next, i.e. if $x_k$ is $k$th intersection, then:
  \[ x_{k+1} = P(x_k). \]
- FP of $P$ correspond to closed orbits
Poincare maps (1)

- Converts problems about closed orbits (tough) into problems about fixed points of a map (easy)
- E.g. stability of limit cycle -> stability of fixed point of Poincare map
- However: typically impossible to find a formula for P
Example

- Poincare map for \( \dot{\Theta} = 1 \)
  \[ \dot{r} = r(1-r^2) \]

- Let S be positive x axis, time of flight for one return is \( t=2\pi \).
  \[
  \int_{r_0}^{r_1} \frac{dr}{r(1-r^2)} = \int_0^{2\pi} dt = 2\pi \\
  r_1 = \left(1 + e^{-4\pi (r_0^2-1)}\right)^{-1/2} \\
  \rightarrow P(r) = \left(1 + e^{-4\pi (r^2-1)}\right)^{-1/2} \\
  \]

- Fixed points? \( r^* = P(r^*) \rightarrow r^* = 1 \)
Cobweb Construction

$r^*$ is stable and unique $\rightarrow$ system has stable limit cycle given by the circle $r=1$. 

![Cobweb Construction Diagram](image)

$r_2 = P(r_1) \
 r_1 = P(r_0) \
 r^* = 1 \quad r$
Linear Stability of Closed Orbits

- Given a system \( \frac{dx}{dt} = f(x) \) with a closed orbit. Is it stable? -> Is corresponding fixed point of Poincare map stable?

- Explore fate of an infinitesimal perturbation \( v_0 \).

\[
x^* + v_1 = P(x^* + v_0) = P(x^*) + [DP(x^*)]v_0 + O(\|v_0\|^2)
\]

\[
\rightarrow v_1 = [DP(x^*)]v_0
\]

- \( DP(x^*) \) is linearized Poincare map at \( x^* \)
Linear Stability of Closed Orbits

• The closed orbit is linearly stable if and only if for all eigenvalues of $DP(x^*)$, $\lambda_i, i=1,\ldots,n-1$ it holds $|\lambda_i|<1$.

$$v_0 = \sum_{j=1}^{n-1} v_{j,0} e_j \quad \rightarrow \quad v_1 = DP(x^*) \sum_{j=1}^{n-1} v_{j,0} e_j = \sum_{j=1}^{n-1} v_{j,0} \lambda_i e_j$$

$$v_k = \ldots = \sum_{j=1}^{n-1} v_{j,0} (\lambda_i)^k e_j$$

• The eigenvalues $\lambda_i$ are also called the characteristic or Floquet multipliers of the closed orbit.

• In general, requires numerical integration ...
Summary

- You should remember:
  - What a limit cycle is and why it is only found in nonlinear systems
  - Conditions under which there cannot be a limit cycle
    - Gradient systems and Liapunov functions
  - The Poincare Bendixson theorem and why it is important
  - Hopf bifurcations
  - Poincare maps and how to argue about stability of closed orbits