Lecture 8

- Fractals and Strange Attractors
  - What is a fractal?
    - Countable and uncountable sets, examples
    - The Cantor set
  - Fractal dimensions
    - Similarity dimension
    - Box dimension
    - Pointwise and correlation dimension
  - Why should strange attractors be fractal?
    - Stretching and folding
    - The pastry map
Introduction

- So far: Claimed that the Lorenz attractor is a fractal, but no definition yet
- Roughly speaking: Fractals are complex geometric shapes with fine structure at arbitrarily small scales
  - Usually they exhibit a degree of self-similarity (magnify tiny part of the whole it has properties that are reminiscent of the whole; often “statistical”)
  - Examples: Clouds, Coastlines, blood vessel networks, broccoli
Countable and Uncountable Sets

- Are some infinities larger than others?
- Cantor: sets X and Y have same cardinality if there exists and invertible mapping from X to Y
- Natural numbers N={1,2,3,...} provides basis for comparisons
  - If X has same cardinality as N, X is countable, otherwise uncountable
- Example: The set E={2,4,6,...} of even numbers is countable
  - Proof: use mapping e(n)=2n
  - Exactly as many numbers as natural numbers!
Countable and Uncountable Sets (1)

- Alternative definition: $X$ is countable if it can be written as a list $\{x_1, x_2, x_3, \ldots\}$ such that for any $x$ there is an $n$ with $x_n = x$.

- Example: the set of integers is countable
  - $\{0, 1, -1, 2, -2, 3, -3, \ldots\} \rightarrow$ any particular integer appears, so set is countable

- Example: the set of positive rational numbers is countable

\[
\begin{array}{cccccc}
1 \\
1 \\
2 \\
1 \\
2 \\
3 \\
1 \\
3 \\
2 \\
3 \\
\vdots
\end{array}
\]
Countable and Uncountable Sets (2)

- Example: The set $X$ of all real numbers between 0 and 1 is uncountable
  - If $X$ were countable we could write it as $X=\{x_1, x_2, x_3, \ldots\}$
  - Write numbers in decimal form
    $$x_1 = 0.x_{11}x_{12}x_{13}x_{14} \ldots$$
    $$x_2 = 0.x_{21}x_{22}x_{23}x_{24} \ldots$$
    $$x_3 = 0.x_{31}x_{32}x_{33}x_{34} \ldots$$
  - Show that number $\bar{x}$ between 0 and 1 is not in the list
    - $1^{\text{st}}$ digit: anything other than $x_{11} \rightarrow \bar{x_{11}} \neq x_{11}$
    - $2^{\text{nd}}$ digit: anything other than $x_{22} \rightarrow \bar{x_{22}} \neq x_{22}$
    - And so on. $\bar{x}=x_{11}x_{22}x_{33} \ldots$ is not in the list!
    - “Diagonal argument”
Cantor Set

- Start with $S_0 = [0,1]$
- Remove the open middle $(1/3, 2/3)$ to obtain $S_1$
- Remove open middle of remaining intervals in $S_1 \rightarrow S_2$
- Repeat ... the limiting set $C = S_\infty$ is the Cantor set
Cantor Set (1)

- Has some properties typical of fractals more generally
  - Structure at arbitrarily fine scales
  - C is self-similar (more generally fractals are only approximately self-similar)
  - C has a non-integer dimension $\ln 2/\ln 3 = 0.63...$
- C has measure zero
  - C is covered by $S_n$. $L_n = (2/3)^n$
- C is uncountable
  - n base 3 expansion C is set of numbers without a “1”
  - Then use diagonal argument
General Cantor Sets

A closed set is a topological Cantor set if:

- S is “totally disconnected”, i.e. S contains no connected subsets (or no intervals in 1d)
- S contains no “isolated points”, i.e. every point in S has a neighbour arbitrarily close by

Cantor sets are spread apart and packed together!

Cross section of strange attractors are often topological Cantor sets but not necessarily self-similar
Dimensions of Self-Similar Fractals

• “Classically”: minimum number of coordinates needed to describe every point in a set

  • -> Paradoxes, like with von Koch curve, which has infinite length and every point is infinitely far away from every other point!

  • K has dimension larger than 1, but no area, so $1<d<2$?
Similarity Dimension

- In 2d: If we scale linear dimension of objects by \( r \) it takes \( m=r^2 \) scaled objects to cover the original object.

- In 3d: \( \rightarrow \) need \( m=r^3 \) scaled objects ...  

- Suppose a self-similar object is composed of \( m \) copies of itself scaled down by a factor \( r \) \( \rightarrow m=r^d \)

- **Similarity dimension** \( d=\ln m/\ln r \)
Similarity Dimension (1)

• What is the similarity dimension of the Cantor set?
  • Need \( m=2 \) objects scaled down by a factor of \( r=3 \) to reproduce the original
  • \( d=\ln 2/ \ln 3! \)

\[
\begin{array}{cccc}
    & 0 & 1/3 & 2/3 & 1 \\
0 & \_ & \_ & \_ & \_ & S_0 \\
1/3 & \_ & \_ & \_ & \_ & S_1 \\
2/3 & \_ & \_ & \_ & \_ & S_2 \\
1 & \_ & \_ & \_ & \_ & S_3 \\
\end{array}
\]
Box Dimension

- Idea: “Measure a set at scale $\varepsilon$” and investigate how measurements vary for $\varepsilon$ to 0.

- $N(\epsilon) \propto L/\epsilon$

- $N(\epsilon) \propto A/\epsilon^2$

- $N(\epsilon) \propto 1/\epsilon^d$

- Box dimension $d = \lim_{\epsilon \to 0} \ln N(\epsilon)/\ln(1/\epsilon)$ (if it exists)
Box Dimension of the Cantor Set

- Set $S_n$ consists of $2^n$ intervals of size $(1/3)^n$
- Pick $\epsilon=(1/3)^n$, need $2^n$ of these intervals to cover $S^n$
  \[ N(\epsilon) = 2^n \quad \text{for } \epsilon=(1/3)^n \]
- Hence:
  \[ d = \lim_{\epsilon \to 0} \frac{\ln N(\epsilon)}{\ln (1/\epsilon)} = \ln 2^n / \ln 3^n = \ln 2 / \ln 3 \]
Box Dimension

• Problems:
  • Not always easy to find minimal covers
    − Alternatively: use mesh of side $\varepsilon$ and count number of occupied boxes $N(\varepsilon)$ in mesh
    − Rarely used in practice: -> storage space and computing time
  • Mathematical problems ...
    − Box dimension of rational numbers is 1 (!)
    − -> Hausdorff dimension
Pairwise and Correlation Dimensions

• Suppose we have a chaotic system that settles on a strange attractor. How to measure its dimension?
• E.g.: sample many points and calculate box dim.
• Better: Grassberger and Procaccia

- \( N_x(\varepsilon) \) ... number of points in \( \varepsilon \)-environment of \( x \)
- Pointwise dimension at \( x \):
  \[ N_x(\varepsilon) \propto \varepsilon^d \]
- Correlation dimension:
  \[ \langle N_x(\varepsilon) \rangle_x \propto \varepsilon^d \]
- Generally: \( d_{\text{correlation}} \leq d_{\text{box}} \) (but not much difference)
Strange Attractors and Cantor Sets

- So far:
  - we know what happens, but not why it happens.
  - E.g.: Why can a differential equation generate a fractal attractor?
- Strange attractors have two properties that seem hard to reconcile
  - Confined to a bounded region in phase space
  - Trajectories separate exponentially fast from neighbours
- How is both possible?!
Stretching and Folding!

● Consider small blob of ICs in phase space
  ● Flow often contracts blob in some direction (dissipation)
  ● And stretches it in the other (exponential separation)
  ● Cannot stretch forever (bounded region), so it must fold back on itself

● Example: pastry map
Hoerseshoes ...

- Limiting set consists of infinitely many smooth layers separated by gaps of varying sizes ...
- ... eventually becomes a Cantor set!
Summary

- Fractals
  - What is it?
  - Countable vs uncountable sets
  - Cantor set construction
  - Fractal dimensions:
    - Similarity dimension
    - Box dimension
    - Pointwise+correlation dimensions
  - Stretching and folding