The Hodgkin–Huxley Model / 1

Current–balance equation:

\[ C_m \frac{dV}{dt} = I_{\text{leak}}(t) + I_{\text{Na}}(t) + I_K(t) + I_{\text{ext}}(t) \]

Voltage–dependent ionic conductances:

\[ g_{\text{Na}}(t) = g_{\text{Na}} \cdot m^3(t) \cdot h(t) \]

\[ g_K(t) = g_K \cdot n^4(t) \]

Individual ionic currents:

\[ I_{\text{leak}}(t) = g_{\text{leak}} \left( E_{\text{leak}} - V(t) \right) \]

\[ I_{\text{Na}}(t) = g_{\text{Na}}(t) \left( E_{\text{Na}} - V(t) \right) \]

\[ I_K(t) = g_K(t) \left( E_K - V(t) \right) \]
The core of the HH model, *voltage dependent* gate kinetics:

\[
\frac{dm}{dt} = \alpha_m(V(t))(1-m(t)) - \beta_m(V(t))m(t) = \frac{m_\infty(V(t)) - m(t)}{\tau_m(V(t))}
\]

\[
m_\infty(V) = \frac{\alpha_m(V)}{\alpha_m(V) + \beta_m(V)}
\]

\[
\tau_m(V) = \frac{1}{\alpha_m(V) + \beta_m(V)}
\]
The Hodgkin–Huxley Model

The HH model working:

- Membrane potential
- Gating variables
- Channel conductances
- Channel currents
Lumping the Variables of the HH Model

A single spike in the phase−space (3 projections)

\[
\begin{align*}
    \frac{dV}{dt} &= -g_{Na} (E_{Na} - V(t)) m'_\infty(t)(1-W(t)) + g_K (E_K - V(t)) \left( \frac{W(t)}{s} \right)^4 + g_{leak} (E_{leak} - V(t)) + I_{ext}(t) \\
    \frac{dW}{dt} &= \frac{W_\infty(V(t)) - W(t)}{\tau_W(V(t))}
\end{align*}
\]
Phase Plane Analysis: the FitzHugh–(Nagumo–, Rinzel–) model

The phase plane with the direction field

\[
\begin{align*}
\frac{dV}{dt} &= 0 \\
\frac{dW}{dt} &= 0
\end{align*}
\]

fixed point

W nullcline

V nullcline
Phase Plane Analysis: the FitzHugh– (Nagumo–, Rinzel–) model / 2

Excitability: firing is a threshold phenomenon (all–or–none)
Phase Plane Analysis: the FitzHugh– (Nagumo–, Rinzel–) model / 3
Phase Plane Analysis: the FitzHugh– (Nagumo–, Rinzel–) model

Post Inhibitory Rebound:
  firing to transient hyperpolarization
The Cable Equation / 1

\[ I_{m}(x,t) = I_{c}(x,t) + I_{\text{leak}}(x,t) + \ldots = -C_{m} \frac{\partial V}{\partial t} - \frac{V(x,t)}{R_{m}} \]

simplification: no voltage-dependent currents!

\[ x \text{ [cm]}, \ t \text{ [msec]} \]
\[ V \text{ [mV]}, I_{\text{axial}} \text{ [\mu A]}, I_{m} \text{ [\mu A/cm]} \]
\[ R_{i} \text{ [k}\Omega/\text{cm}], R_{m} \text{ [k}\Omega/\text{cm}], C_{m} \text{ [\mu F/cm]} \]
\[ \lambda = \sqrt{R_{m}/R_{i}} \text{ [cm]} \]
\[ \tau = R_{m}C_{m} \text{ [msec]} \]

\[ \frac{1}{R_{i}} \frac{\partial^{2} V}{\partial x^{2}} - \frac{1}{C_{m}} \frac{\partial V}{\partial t} - \frac{V(x,t)}{R_{m}} = 0 \]

\[ \lambda^{2} \frac{\partial^{2} V}{\partial x^{2}} - \tau \frac{\partial V}{\partial t} - V(x,t) = 0 \]
The Cable Equation / 2

Constant current injection: steady-state **spatial** voltage spread

Transient current injection: **temporal** development of voltage spread
Multicompartmental modeling

\[
\frac{1}{R_i} \frac{\partial^2 V}{\partial x^2} - C \frac{\partial V}{\partial t} - \frac{V(x,t)}{R_m} = 0
\]

\[
C_k \frac{dV_k}{dt} = I_k(t) + \frac{V_{k-1}(t) - V_k(t)}{R_k} + \frac{V_{k+1}(t) - V_k(t)}{R_{k+1}}
\]

all sorts of ionic currents (HH, etc)