Bifurcation Phenomena of Simple Pulse-Coupled Spiking Neuron Models with Filtered Base Signal

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Abstract—This paper studies nonlinear phenomena of pulse-coupled bifurcating neurons. Repeating integrate-and-fire dynamics between a constant threshold and periodic base signal, the bifurcating neuron outputs a spike-train. Applying a low-pass filter to the periodic square wave, we obtain the base signal. As parameters of the filter vary, the shape of the base signal varies and the neurons can exhibit a variety of periodic/chaotic spike-trains and related bifurcation phenomena. We consider typical phenomena. First, the single bifurcating neuron can exhibit the period doubling bifurcation where both period and the number of spike-trains are doubling. Second, the pulse-coupled two bifurcating neurons can exhibit the tangent bifurcation that causes "chaos + chaos = order": chaotic spike-trains of two single neurons are changed into periodic spike-train by the pulse-coupling. Such phenomena are filter-induced bifurcations because they are caused by the filtering. The bifurcation sets are calculated precisely based on the state equation of the filter and the one-dimensional spike-phase map. Presenting a simple test circuit, typical phenomena are confirmed experimentally.

I. INTRODUCTION

The bifurcating neuron (BN, [1]-[4]) is a simple switched dynamical system inspired by spiking neuron models [5]-[7]. Repeating integrate-and-fire dynamics between a constant threshold and periodic base signal, the BN can output a variety of periodic spike-trains (PSTs) and chaotic spike-trains. The BN can be a building block of pulse-coupled systems that can exhibit a variety of synchronous/asynchronous phenomena [8]-[10]. Motivations for studying such spike-based dynamical systems are many, including the following. Classification and analysis of the nonlinear phenomena are important as basic study of nonlinear dynamical systems [11]. The results of the analysis can be basic information to consider modeling and learning of neural systems [7] [12]-[14]. They can be basic information also for the real/potential engineering applications including image/signal processing [8] [9], spike-based communications [15], analog-to-digital conversion [16], neural prosthesis [17], spatio-motor transformations [18], and reservoir computing [14].

This paper studies dynamics of the single BN and pulse-coupled two bifurcating neurons (PCBN) with filtered base signal. The contents include the following key points. First, the dynamics of the BN depends crucially on the shape of base signal and the base signal is given by applying basic low-pass filter (LPF) to the periodic square wave. As the parameters of the LPF vary, the shape of the base signal varies and the BN can output a variety of spike-trains. The LPF is simple in implementation and is suitable for laboratory experiments.

Second, in the analysis, the periodic base signal is described by a steady state solution of the state equation of the LPF. This description enables us to realize precise analysis with low computation cost. The dynamics of the BN and PCBN can be analyzed precisely by one-dimensional map of spike-phases (Pmap). Such precise analysis is impossible in the Fourier series approximation of the base signal (e.g., the Gibbs' phenomenon occurs for discontinuous signal).

Third, as the parameters of the LPF vary, the BN and PCBN exhibit a variety of bifurcation phenomena and typical phenomena are discussed in this paper. The single BN exhibits period-doubling bifurcation where both the period and the number of PSTs are doubling. The doubling continues and the PST is changed into chaotic spike-train. The PCBN exhibits two kinds of tangent bifurcations (smooth and non-smooth ones) where chaotic spike-train of the PCBN is changed into a PST. It can cause "chaos + chaos = order": chaotic spike-trains of two single BNs are changed into a PST by the pulse-coupling. We refer to such phenomena as filter-induced bifurcations because the BN and PCBN cannot exhibit chaos and bifurcation if the filter does not present (the base signal is periodic square wave). Presenting a simple test circuit, typical phenomena are confirmed experimentally.

Analysis of the filter-induced bifurcation is important not only as a basic study but also to consider signal processing function of neuron systems, because filtering effects are inevitable in signal processing in real world. Note that the parameters of the LPF (e.g., time constant) have practical meaning from viewpoint of signal processing.

In order to clarify the novelty of this paper, we should comment on several previous works. Refs. [1]-[4] [19]-[21] study BN and PCBN where the base signal is given without filtering (e.g., pure sinusoidal base signal): no filter-induced bifurcation is discussed. Ref. [22] studies BN and PCBN whose base signal is given by filtering the triangular waveform. However, it uses the Fourier series approximation of the base signal and discusses neither precise analysis based on the state equation nor precise calculation of the bifurcation sets. The triangular base signal can cause chaos without filtering: it is different from the filter-induced chaos in this paper. Also, these previous papers do not discuss doubling of the number of PSTs in the period doubling bifurcation.
II. BIFURCATING NEURONS

Figure 1 illustrates the dynamics of the BN. The state variable \( x \) rises with slope \( s \) below the threshold \( x = 1 \). When \( x \) reaches the threshold, the BN outputs a spike \( y = 1 \) and \( x \) is reset to the base signal \( b(\tau) \) instantaneously where \( \tau \) is the dimensionless time. Repeating in this manner, the BN outputs a spike-train \( y(\tau) \). The dynamics is described by

\[
\begin{aligned}
x(\tau + 1) &= b(\tau), \\
y(\tau + 1) &= 1 \\
\end{aligned}
\]

where the base signal is period 1, \( b(\tau + 1) = b(\tau) \), and \( b(\tau) < 1 \). A simple circuit implementation of the BN is discussed in Section VI. The spike-train \( y(\tau) \) is characterized by spike positions. Let \( \tau(n) \) denote the \( n \)-th spike position. Since \( \tau(n + 1) \) is determined by \( \tau(n) \), we can define the spike-position map (Smap):

\[
\tau(n + 1) = \tau(n) - (b(\tau(n)) - 1)/s \equiv F(\tau(n))
\]

Since \( F(\tau + 1) = F(\tau) + 1 \) is satisfied, we introduce the phase variable \( \theta(n) = \tau(n) \mod 1 \). Using this, we can define the spike-phase map (Pmap):

\[
\theta(n + 1) = f(\theta(n)) \equiv F(\theta(n)) \mod 1
\]

Using the Pmap, we can analyze the dynamics. Note that the dynamics crucially depends on the shape of \( b(\tau) \). Although there are many methods to make various shapes of \( b(\tau) \), we apply a basic Low-pass filter (LPF) to the following periodic square wave

\[
b_s(\tau) = \begin{cases} 
-a & \text{for } 0 \leq \tau < 0.5 \\
a & \text{for } 0.5 \leq \tau < 1
\end{cases}
\]

where \( b_s(\tau + 1) = b_s(\tau) \) and \( 0 < a < 1 \). Figure 2 shows an example of the Smap and Pmap for the square base signal \( b(\tau) = b_s(\tau) \) ( before applying the LPF). In this case, \( Df(\theta) = 1 \) for almost all \( \theta \) and the BN cannot exhibit chaos and related bifurcation. Note that at points \( \tau(n) = 0.5m \) ( \( m \) denotes integers) the Smap is discontinuous when \( b(\tau) = b_s(\tau) \) and the derivative of Smap is discontinuous when the LPF is applied to \( b_s(\tau) \). We refer to \( \tau(n) = 0.5m \) as breakpoints of the Smap and refer to \( \theta(n) = 0.5 \) and 1 as breakpoints of the Pmap. The breakpoints play an important role in the tangent bifurcation in Section ???. Applying the LPF, the shape of \( b(\tau) \) varies and the BN can exhibit various phenomena. In this paper, we describe the filtered base signal by the equation:

\[
\lambda x_c = - x_c + b_s(\tau)
\]

This is the dimensionless state equation of the LPF (RC circuit) shown in Section VI and \( x_c \) is the state variable ( the output of the LPF). The steady state solution \( (x_c) \) corresponds to the filtered base signal \( b_s(\tau) \) that is given as the following after simple calculation:

\[
b_s(\tau) = \begin{cases} 
(x_0 + a)e^{-\tau/\lambda} - a & \text{for } 0 \leq \tau < 0.5 \\
-(x_0 + a)e^{-(\tau - 0.5)/\lambda} + a & \text{for } 0.5 \leq \tau < 1
\end{cases}
\]

Figure 2. Spike-position map (Smap) and spike-phase map (Pmap) for square wave base signal with \( a = 0.7 \).

The state variable \( x \) rises with slope \( s \) below the threshold \( x = 1 \). When \( x \) reaches the threshold, the BN outputs a spike \( y = 1 \) and \( x \) is reset to the base signal \( b(\tau) \) instantaneously where \( \tau \) is the dimensionless time. Repeating in this manner, the BN outputs a spike-train \( y(\tau) \). The dynamics is described by

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\begin{aligned}
\dot{x} &= s, \\
y &= 0 \quad \text{for } x < 1 \\
x(\tau + 1) &= b(\tau), \\
y(\tau + 1) &= 1 \quad \text{for } x(\tau) \geq 1
\end{aligned}
\]

where the base signal is period 1, \( b(\tau + 1) = b(\tau) \), and \( b(\tau) < 1 \). A simple circuit implementation of the BN is discussed in Section VI. The spike-train \( y(\tau) \) is characterized by spike positions. Let \( \tau(n) \) denote the \( n \)-th spike position. Since \( \tau(n + 1) \) is determined by \( \tau(n) \), we can define the spike-position map (Smap):

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where \( b_s(\tau + 1) = b_s(\tau) \) and \( 0 < a < 1 \). Figure 2 shows an example of the Smap and Pmap for the square base signal \( b(\tau) = b_s(\tau) \) ( before applying the LPF). In this case, \( Df(\theta) = 1 \) for almost all \( \theta \) and the BN cannot exhibit chaos and related bifurcation. Note that at points \( \tau(n) = 0.5m \) ( \( m \) denotes integers) the Smap is discontinuous when \( b(\tau) = b_s(\tau) \) and the derivative of Smap is discontinuous when the LPF is applied to \( b_s(\tau) \). We refer to \( \tau(n) = 0.5m \) as breakpoints of the Smap and refer to \( \theta(n) = 0.5 \) and 1 as breakpoints of the Pmap. The breakpoints play an important role in the tangent bifurcation in Section ???. Applying the LPF, the shape of \( b(\tau) \) varies and the BN can exhibit various phenomena. In this paper, we describe the filtered base signal by the equation:

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This is the dimensionless state equation of the LPF (RC circuit) shown in Section VI and \( x_c \) is the state variable ( the output of the LPF). The steady state solution \( (x_c) \) corresponds to the filtered base signal \( b_s(\tau) \) that is given as the following after simple calculation:

\[
b_s(\tau) = \begin{cases} 
(x_0 + a)e^{-\tau/\lambda} - a & \text{for } 0 \leq \tau < 0.5 \\
-(x_0 + a)e^{-(\tau - 0.5)/\lambda} + a & \text{for } 0.5 \leq \tau < 1
\end{cases}
\]

where \( x_0 \) is the initial value for the steady state solution. The parameter \( \lambda \) is used as a subscription of the base signal hereafter. For \( \lambda > 0 \), stability of the steady state solution can be confirmed easily. The \( \lambda \) characterizes effects of the LPF and is proportional to the time constant of the basic LPF in Section VI. Substituting Eq. (6) into Eqs. (3) and (2), we obtain

\[
\theta(n + 1) = f(\theta(n)) \equiv F(\theta(n)) \mod 1
\]

Using this formula, we can analyze the dynamics precisely. If we use the Fourier series approximation for the base signal, such a precise analysis is impossible. For simplicity, we select \( \lambda \) as the control parameter and fix \( (s, a) = (1, 0.7) \) after trial-and-errors. We use this one control parameter for the single BN and use two control parameters for the PCBN.

In order to analyze the dynamics of BN, we give basic definitions for the Pmap. A point \( p \) is said to be a periodic point (PEP) with period \( k \) if \( f^k(p) = p \) and \( f^l(p) \neq p \) for \( 0 < l < k \) where \( f^k \) is the \( k \)-fold composition of \( f \). A PEP with period 1 is referred to as a fixed point. Let \( Df^k(\theta) \) denote derivative of \( f^k \) by \( \theta \) at \( p \). A PEP with period \( k \) is said to be stable, critical and unstable for initial value if \( |Df^k(\theta)| < 1 \), \( |Df^k(\theta)| = 1 \) and \( |Df^k(\theta)| > 1 \), respectively. A sequence of PEPs \( \{ f(p), f^2(p), \ldots, f^k(p) \} \) is said to be a periodic orbit (PEO) with period \( k \). A PEP with period \( k \) corresponds to one periodic spike-train (PST) and a PEO with period \( k \) corresponds to \( k \) PSTs.
We introduce the Lyapunov exponent of the Pmap.

\[
\Lambda = \frac{1}{M} \sum_{n=1}^{M} \ln |Df(x_n)| \tag{8}
\]

where \( M \) is a large number. The \( Df \) is calculated by

\[
D f(\theta) = 1 - \frac{d}{d\theta} h_\lambda(\theta) = \begin{cases} 
1 + \frac{x_0 + a}{\lambda} e^{-\theta/\lambda} & \text{for } 0 \leq \theta < 0.5 \\
1 - \frac{x_0 + a}{\lambda} e^{-(\theta - 0.5)/\lambda} & \text{for } 0.5 \leq \theta < 1 
\end{cases} \tag{9}
\]

Note that \( Df \) is discontinuous at the breakpoints \( \theta = 0.5 \) and 1 (the endpoints 0 and 1 are connected if we regard \( f \) as a map on the unit circle just like the well-known circle map \([11]\)). As is well known, \( \Lambda > 0 \) is used as an indicator of chaotic behavior and \( \Lambda < 0 \) guarantees stability of PEOs.

### III. Period Doubling Bifurcation of the BN

Figure 3 shows typical phenomena in the Pmap. For \( \lambda = 0.5 \), the Pmap has stable fixed point (\( \alpha \) in Fig. 3 (a)). As \( \lambda \) decreases, the fixed point becomes unstable and stable PEO with period 2 by the period doubling bifurcation. Note that the stable fixed point corresponds to one PST as shown in Fig. 3 (a') and the stable PEO with period 2 corresponds to two PSTs as shown in Fig. 3 (b'). That is, in the BN and Pmap, the period doubling bifurcation implies not only doubling of period but also doubling of the number of PSTs. The plural PSTs can co-exist and the BN exhibits either PST or chaotic behavior. Hence we refer this phenomenon as the filter-induced bifurcation.

Figure 4 shows the bifurcation diagram and Lyapunov exponent. Convergence of \( \Lambda \) is confirmed for \( M > 10000 \). At \( \lambda = \lambda_p \), the first period doubling occurs where \( D f(\alpha) = -1 \). At \( \lambda = \lambda_b \), the stable PEO with period 2 collides with the breakpoint \( \theta(n) = 1 \) (see Fig. 3 (c)) and \( \Lambda \) of the PEO is discontinuous because of the discontinuity of \( Df \) at the break point. This is a kind of the border-collision bifurcation. The parameter value \( \lambda = \lambda_b \) is a key point in the tangent bifurcation of the PCBN in Section V. For \( \lambda < \lambda_c \), \( \Lambda \) becomes positive and chaotic behavior appears.
IV. PULSE-COUPLED BIFURCATING NEURONS

Applying the cross-coupling to two bifurcating neurons BN1 and BN2, we construct the PCBN. The cross-coupling is defined as the following (see Fig. 5). If \( x_1 \) raises and reaches the threshold \( x_1 = 1 \) then BN1 output a spike \( y_1 \) that reset \( x_2 \) to the base \( b_{\lambda_1}(\tau) \) instantaneously. In a likewise manner, if \( x_2 \) raises and reaches the threshold \( x_2 = 1 \) then BN2 output a spike \( y_2 \) that reset \( x_1 \) to the base \( b_{\lambda_2}(\tau) \) instantaneously. The dynamics is described by

\[
\begin{align*}
\dot{x}_1 &= s_1, & y_2 &= 0 & \text{for } x_2 < 1 \\
\dot{x}_1(\tau_+) &= b_{\lambda_1}(\tau_+), & y_2(\tau_+) &= 1 & \text{if } x_2(\tau) \geq 1 \\
\dot{x}_2 &= s_2, & y_1 &= 0 & \text{for } x_1 < 1 \\
\dot{x}_2(\tau_+) &= b_{\lambda_2}(\tau_+), & y_1(\tau_+) &= 1 & \text{if } x_1(\tau) \geq 1 \\
\end{align*}
\]

where the base signals \( b_{\lambda_1}(\tau) \) and \( b_{\lambda_2}(\tau) \) are given by applying the LPF to the square wave of Eq. (4) with amplitude \( a_1 \) and \( a_2 \), respectively (and are calculated by the state equation). As shown in Fig. 5, let \( \tau_1(n) \) be the \( n \)-th spike position of BN1 just after \( x_1 \) reaches 1 and let \( \tau_2(n+1) \) be the \( (n+1) \)-th spike position of BN2 just after \( x_2 \) reaches 1. Following the definition of the cross-coupling, \( \tau_1(n) \) determines \( \tau_2(n+1), \tau_2(n+1) \) determines \( \tau_1(n+2) \) and the spike-positions are described by

\[
\begin{align*}
\tau_2(n+1) &= F_1(\tau_1(n)), \quad \tau_1(n+2) = F_2(\tau_2(n+1)) \\
F_1(\tau) &\equiv \tau - (b_{\lambda_1}(\tau) - 1)/s_1 \\
F_2(\tau) &\equiv \tau - (b_{\lambda_2}(\tau) - 1)/s_2
\end{align*}
\]

(12)

where \( F_1 \) and \( F_2 \) are Smaps of BN1 and BN2 with self-firing, respectively. If the BN1 (BN2) exhibits the first firing at \( n = 1 \) then the Smap of the PCBN is given by Eq. (12) with odd \( n \) (even \( n \)). For simplicity, we consider the case of odd \( n \) hereafter:

\[
\tau_1(n+2) = F_2(F_1(\tau_1(n))) \equiv F_{21}(\tau_1(n)) \quad \text{for odd } n
\]

That is, the Smap \( F_{21} \) of the PCBN is given by composition of Smaps \( F_1 \) and \( F_2 \) of single BNs. Let \( \theta_i(n) = \tau_i(n) \mod 1, i = 1, 2 \), be the phase variables. Using these, we can define the Pmap of the PCBN:

\[
\begin{align*}
\theta_1(n+1) &= f_2(f_1(\theta_1(n))) \equiv f_{21}(\theta_1(n)) \\
f_1(\theta) &\equiv F_1(\theta) \mod 1, \quad f_2(\theta) &\equiv F_2(\theta) \mod 1
\end{align*}
\]

(13)

That is, the Pmap \( f_{21} \) of the PCBN is given by composition of Pmaps \( f_1 \) and \( f_2 \) of single BNs. For simplicity, we fix

\[ s_1 = s_2 = 1, \quad a_1 = a_2 = 0.7 \]

and select \( \lambda_1 > 0 \) and \( \lambda_2 > 0 \) as the control parameters.

V. TANGENT BIFURCATION OF THE PCBN

Here we consider two kinds of tangent bifurcations of PCBN through which chaotic spike-train is changed into PSTs. As is well known, intermittent chaotic behavior appears near this bifurcation [11]. The first tangent bifurcation set is characterized by

\[
\text{TB1} = \{(\lambda_1, \lambda_2) | f_{21}(p_1) = p_1, \quad Df_{21}(p_1) = 1\}
\]

(14)

where \( p_1 \) is a fixed point near the local minimum as shown in \( I_a \) in Fig. 6 (a). On the TB1, the fixed point \( p_1 \) touches the line \( \theta_1(n+1) = \theta_1(n) \) at which \( Df_{21}(p_1) = 1 \) and \( Df_{21} \) is continuous (\( f_{21} \) is smooth) as shown in Fig. 6 (a'). The fixed point \( p_1 \) is a solution of Equation

\[
f_2(f_1(p_1)) = p_1.
\]

(15)

It can be calculated precisely by Eqs. (7) and (11). The derivative \( Df_{21}(p_1) = 1 \) can be calculated by substituting the solution of Eq. (15) into

\[
Df_{21}(p_1) = Df_2(f_1(p_1))Df_1(p_1) = 1
\]

where

\[
Df_i(\theta) = 1 - \frac{df_i(\theta)}{d\theta}, \quad i = 1, 2
\]

(16)

After the calculation, we obtain TB1 as shown in Fig. 7. The second tangent bifurcation set is characterized by

\[
\text{TB2} = \{(\lambda_1, \lambda_2) | f_{21}(p_2) = p_2, \quad f_{21}(p_2) = 1\}
\]

(17)

where \( p_2 \) is a breakpoint of \( f_{21} \). At TB2, the breakpoint \( p_2 \) becomes the fixed point at which the Pmap \( f_{21} \) touches the line \( \theta_1(n+1) = \theta_1(n) \) and \( Df_{21} \) is discontinuous (\( f_{21} \) is nonsmooth) as shown in Figs. 6 (b) to (c'): \( 0 < Df_{21+}(p_2) < 1 \) and \( 1 < Df_{21-} \) are the left- and right-derivative of \( f_{21} \), respectively. The TB2 is a solution curve of Equation

\[
f_2(f_1(p_2)) = p_2, \quad f_1(p_2) = 1
\]

Using the Pmap, we can calculate the TB2 as shown in Fig. 7. Since exchanging \( \lambda_1 \) and \( \lambda_2 \) implies exchanging BN1 and BN2, the bifurcation sets are symmetric with respect to the line \( L_s \equiv \{(\lambda_1, \lambda_2) | \lambda_1 = \lambda_2\} \). On the line \( L_s \), the Pmap
is $f_{21} = f_{2}^2 = f_{2}^2$ and the dynamics is equivalent to the single BN. For $\lambda_2 > \lambda_1$, as $\lambda_1$ increases, the TB1 is changed into TB2. As shown in Fig. 6 (b), at the border between TB1 and TB2, the Pmap $f$ becomes non-smooth at the fixed point $p_2$ such that $Df_{21}(p_2) = 1$ and $Df_{21+}(p_2) > 1$. As shown in Fig. 7, the TB2 intersects with the line $L_4$ and the intersection is $\lambda_1 = \lambda_2 = \lambda_0$. At $\lambda = \lambda_0$, the single BN causes the border-collision bifurcation. In summary, in the parameter space, we can say (1) The PCBN has two kinds of tangent bifurcation sets (smooth TB1 and non-smooth TB2), (2) The TB1 connects to TB2, and (3) The end point of the TB2 corresponds to the border-collision bifurcation set of single BN. It should be noted that (1) is impossible in the case of sinusoidal or triangular base signal in Refs. [19] [20] because the smooth base signal can cause the smooth tangent bifurcation only and the non-smooth base signal can cause the non-smooth tangent bifurcation only.

The tangent bifurcation causes stable fixed point (see Figs. 8 (a) and 9 (c')) that corresponds to a stable PST of the PCBN (see Fig. 10 (c)). As shown in the bifurcation diagram in Fig. 8, as $\lambda_1$ increases, chaotic orbit (spike-train) is changed into PEOs via the TB1 and then to chaotic orbit. Here, it should be noted that, as shown in Fig. 4, single BNs exhibit chaotic spike-train in some parameter sub-space for $\lambda < \lambda_c$. 

![Fig. 6. Tangent bifurcation for ($a_1 = a_2 = 0.7$). (a) & (a') Pmap of TB1 for ($\lambda_1, \lambda_2$) = (0.055, 0.1) and enlargement of $I_a$. (b) & (b') Pmap of on the border between TB1 and TB2 for ($\lambda_1, \lambda_2$) = (0.1169, 0.163) and enlargement of $I_a$. (c) & (c') Pmap of TB2 for ($\lambda_1, \lambda_2$) = (0.2638, 0.27) and enlargement of region $I_c$. (d) & (d') Pmap of at the end of TB2 for ($\lambda_1, \lambda_2$) = (0.279, 0.279) and enlargement of $I_d$.](image1)

![Fig. 7. Tangent bifurcation sets. Pmaps (a) to (d) of Fig. 6 are given at points a to d, respectively.](image2)

![Fig. 8. Bifurcation for $\lambda_1$ ($\lambda_2 = 0.1$, $a_1 = a_2 = 0.7$). (a) Bifurcation diagram ($\lambda_1 = 0.07 \equiv \lambda_a$ corresponds to Figure 9 (c)). (b) Lyapunov exponent.](image3)
(e.g., near $\lambda_1 = \lambda_0$ in Fig. 8). It means that chaotic spike-trains of two BNs are changed into PST of PCBN by the cross-coupling and the filter-induced TB1. We refer to this phenomenon as "chaos + chaos = order". In the parameter space in Fig. 7, the "chaos + chaos = order" can be found in the region surrounded by TB1, $\lambda_1$-axis, $\lambda_2$-axis, $\lambda_1 = \lambda$ and $\lambda_2 = \lambda_c$. Figs. 9 and 10 show typical Pmaps and corresponding time-domain waveforms of the "chaos + chaos = order".

VI. LABORATORY EXPERIMENTS

Figure 11 shows a circuit model of the PCBN where $r_{ij}$ $(i, j = 1, 2)$ denotes inner resistors of the current/voltage sources. The base signal $B_1(t)$ of BN1 and $B_2(t)$ of BN2 are outputs of basic LPF (RC circuit) whose input is periodic square wave $B_s(t)$ with period $T$. Below the threshold $V_{T_1}$ (respectively, $V_{T_2}$), the capacitor voltage $v_1$ ($v_2$) increases by integrating the current source $I_1$ ($I_2$). If $v_1$ (respectively, $v_2$) reaches $V_{T_1}$ ($V_{T_2}$) then the comparator triggers a monostable multi-vibrator (MM) to output a spike $Y(t)$. The spike closes a switch $S_2$ ($S_1$) and $v_2$ ($v_1$) is reset to the base signal $B_2(t)$ ($B_1(t)$). Repeating in this manner, the PCBN generates a spike-train. The dynamics is described by

$$
C \frac{dv_i}{dt} = \left\{ \begin{array}{ll}
-\frac{1}{\tau_i}v_i + J_i & \text{for } S_i = \text{off} \\
-\frac{1}{\tau_{ii}}v_i + \frac{1}{\tau_{ij}}v_j + \frac{1}{\tau_{ij}}B_s(t) & \text{for } S_i = \text{on}
\end{array} \right.
$$

(18)

$$
RC_i \frac{dv_{ci}(t)}{dt} + v_{ci}(t) = B_i(t)
$$

(19)

where $B_i(t + T) = B_i(t)$, $i = 1, 2$ and $j = 2, 1$. The base signals $B_1(t)$ and $B_2(t)$ are given by the steady state solution of Eq. (19) for $i = 1$ and 2, respectively. In order to simplify Eq. (18), we omit the inner resistors $(r_{ii} \rightarrow \infty, r_{ij} \rightarrow 0)$ and approximate the current source by $I_i \approx J_i(1 - \exp^{-1})$ (the slope of line connecting $(0, 0)$ and $(r_{ii}C, v_i(r_{ii}C)$) of Eq. (18)). Then we have

$$
\left\{ \begin{array}{ll}
C \frac{dv_1}{dt} = I_1 & \text{for } v_2(t) < V_{T_2} \\
v_1(t+) = B_1(t+) & \text{if } v_2(t) = V_{T_2}
\end{array} \right.
$$

(20)

$$
\left\{ \begin{array}{ll}
C \frac{dv_2}{dt} = I_2 & \text{for } v_1(t) < V_{T_1} \\
v_2(t+) = B_2(t+) & \text{if } v_1(t) = V_{T_1}
\end{array} \right.
$$

This equation is transformed into Eqs. (10) and (5) using the following dimensionless variables and parameters:

$$
\tau = \frac{t}{T}, \quad x_i = \frac{v_i}{V_{T_i}}, \quad s_i = \frac{I_i}{CV_{T_i}}, \quad b_{\lambda_i}(\tau) = \frac{B_i(T\tau)}{V_{T_i}}, \\
a_i = \frac{A_i}{V_{T_i}}, \quad \lambda_i = \frac{R_iC_i}{T}, \quad x_c = \frac{v_{ci}}{V_{T_i}}, \quad i = 1, 2.
$$

(21)
We have fabricated the breadboard prototype and have confirmed typical phenomena. For example, we have confirmed "chaos + chaos = order" observed in the laboratory. The BN1 and BN2 exhibit chaotic waveforms before the coupling as shown in Fig. 12 (a) and (b), respectively. Applying the cross-coupling, these are changed into periodic waveform as shown in Fig. 12 (c).

VII. CONCLUSIONS

Filter-induced bifurcation phenomena of the BN and PCBN have been studied in this paper. The base signal is given by applying the LPF to the periodic square wave. As parameters of the LPF vary, the BN and PCBN can exhibit various bifurcation phenomena and typical ones are considered. The BN exhibits period-doubling bifurcation where both the period and the number of PSTs are doubling and the PSTs are changed into chaotic spike-trains. The PCBN exhibits the smooth and non-smooth tangent bifurcations where chaotic spike-train is changed into PSTs. This bifurcation causes the "chaos + chaos = order".

Future problems include detailed analysis of bifurcation phenomena, application of various filters, study of large-scale PCBNs and applications to spike-based engineering systems.

REFERENCES


