A Fast Analogue $K$-Winners-Take-All Neural Circuit

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Abstract — A continuous-time model of analogue $K$-winners-take-all (KWTA) neural circuit which is capable of identifying the $K$ largest of any unknown finite value $N$ distinct inputs, where $I \leq K < N$ is presented. The model is described by a state equation and an output equation. A corresponding functional block diagram of the circuit is presented as $N$ feed-forward hard-limiting neurons and two feedback neurons, which are used to determine the dynamic shift of inputs. The circuit combines such properties as high accuracy, low hardware implementation complexity, and independency of initial conditions. Simulation examples demonstrating the circuit performance and some of its applications are presented. According to simulation results, the circuit possesses a higher convergence speed to the KWTA operation than other comparable analogs.

I. INTRODUCTION

$K$-winners-take-all (KWTA) neural networks are known to choose $K$ largest out of a set of $N$ inputs, where $I \leq K < N$ is a positive integer [1]. In the special case when $K$ is equal to unity, the KWTA network is the winner-takes-all (WTA) one, that determines the maximal out of $N$ inputs [2], [3].

KWTA neural networks have various applications, particularly, in data and signal processing, in decision making, for pattern recognition, in competitive learning, and in sorting [4] - [6]. The KWTA networks are used in telecommunications [7] and vision systems [8], for solving problems of filtering [9], decoding [10], image processing [11], clustering [12], and classifying [13], [14]. The KWTA operation is used in machine learning, in mobile robot navigation, and in feature extraction [15], [16]. The KWTA mechanisms are used for modeling cognitive phenomena and spiking neural networks [17], [18].

Continuous-time KWTA neural networks implemented in analogue hardware can be faster, more compact and more power-efficient compared to digital implementations [19]. Many different analogue neural networks have been proposed to solve the KWTA problem [1], [5], [20] - [22]. In particular, a continuous-time model of the KWTA neural circuit which can select the $K$ maximal out of $N$ unknown inputs, where $I \leq K < N$, located in a definite range of change was proposed in [21]. The operation of the model depends on the initial values of the state variable. Recently, a modification of this model was derived and simulated [23]. In contrast to the predecessor, the modified model is independent of initial conditions and uses a simplified residual function. Computer simulations showed that the model convergence speed to the KWTA operation is close to that of one of the fastest Hopfield type analogue KWTA neural networks, whereas a computational and hardware implementation complexity of the model is lower than the complexity of this network. The hardware implementation complexity of the model is close to that of one of the simplest continuous-time KWTA models, whereas the convergence time to the KWTA operation of the model is lower than that of this comparable model. A discrete-time version of the model and a corresponding functional block-diagram of a digital neural circuit have been proposed in [24].

In this paper, a generalized continuous-time model and a corresponding functional block-diagram of analogue KWTA neural circuit are presented. In contrast to the predecessors proposed in [21], [23], the circuit is capable of selecting $K$ largest out of $N$ unknown inputs, where $I \leq K < N$, located in an unknown range of change. The circuit is described by a differential equation and an output equation. Computer simulations show that the circuit demonstrates a higher convergence speed to the KWTA operation than other comparable analogs. Some applications of the circuit are presented.

II. CONTINUOUS-TIME MODEL OF THE CIRCUIT

Let us generalize a continuous-time model of an analogue KWTA neural circuit presented in [23] based on the case of identifying $K$ maximal out of $N$ unknown inputs, where $I \leq K < N$, located in an unknown range of change. We assume that there is an input vector $a = (a_{n_1}, a_{n_2}, \ldots, a_{n_N}) \in \mathbb{R}^n$, $I < N < \infty$ with unknown finite value elements, the inputs are distinct and can be arranged in a descending order of magnitude satisfying the inequalities

$$\infty > a_{n_1} > a_{n_2} > \ldots > a_{n_N} > -\infty, \quad (1)$$

where $n_1, n_2, \ldots, n_N$ are the unknown numbers of the first largest input, the second largest input and so on up to the $N$-th largest input inclusive. Let us design a continuous-time model of an analogue neural circuit that should identify the $K$ largest of these inputs, which are referred to as the winners. The designed model should process the input vector $a$ to obtain, after a finite convergence time, such an output vector $b = (b_{n_1}, b_{n_2}, \ldots, b_{n_N})$ that the following KWTA property is satisfied:

$$b_{n_i} > 0, i = 1, 2, \ldots, K; \quad b_{n_j} < 0, j = K + 1, K + 2, \ldots, N. \quad (2)$$
We assume that the outputs of the model are given by
\[
\begin{align*}
    b_{ni} &= a_{ni} - x > 0, \quad i = 1, 2, \ldots, K; \\
    b_{nj} &= a_{nj} - x < 0, \quad j = K + 1, K + 2, \ldots, N,
\end{align*}
\]  
(3)

where \( x \) is a scalar dynamic shift of inputs [21].

Let us describe the model of a designed KWTA neural circuit by the following state equation:
\[
\frac{dx}{dt} = \alpha |x| + p \left( \sum_{k=1}^{N} S_k(x) - K \right),
\]
(4)

and an output equation
\[
b_{nk} = a_{nk} - x, \quad k = 1, 2, \ldots, N,
\]
(5)

where
\[
R(x) = \sum_{k=1}^{N} S_k(x) - K
\]
(6)
is a residual function,
\[
S_k(x) = \begin{cases} 
1, & \text{if } a_{nk} - x > 0; \\
0, & \text{otherwise}
\end{cases}
\]
(7)
is a step function, \( \sum_{k=1}^{N} S_k(x) \) is the number of positive outputs of the model, \( \alpha \) is the gain which can be used to control a convergence speed of the model state variable trajectories to the KWTA operation, \( p \) is a constant parameter, \( -\infty < x_0 < \infty \) is an initial condition. Note that the state equation (4) can also be transformed to the following special form:
\[
\frac{dx}{dt} = \gamma |x + c| \text{sgn}(R(x))
\]
(8)

where
\[
\text{sgn}(R(x)) = \begin{cases} 
1, & \text{if } R(x) > 0; \\
0, & \text{if } R(x) = 0; \\
-1, & \text{if } R(x) < 0
\end{cases}
\]
(9)
is a signum (hard-limiting) function, \( \gamma \) is a gain, and \( c \) is a constant parameter.

III. ANALOGUE FUNCTIONAL BLOCK-DIAGRAM OF THE CIRCUIT

The functional block-diagram of a generalized analogue KWTA neural circuit (GA circuit) built based on the model described by the state equation (4) and output equation (5) is shown in Fig. 1. The diagram consists of inputs \( a_1, a_2, \ldots, a_N \), summers \( \Sigma \), an integrator \( I \) with the gain \( \alpha \),

external sources of constant signals \( K, x_0, p \), blocks \( S_1, S_2, \ldots, S_N \) of step functions \( S_k(x) \), \( k = 1, 2, 3, \ldots, N \), outputs \( b_1, b_2, \ldots, b_N \), block of multiplication \( \times \), and a block of module function \( \text{Abs} \). Note that the outputs of blocks \( S_1, S_2, \ldots, S_N \) can also be used as the circuit outputs. However, in this case, the only \( K \) winners out of \( N \) inputs will be identified. No information is available on the ordering of inputs by magnitude which can be useful for some applications [25].

As we can see, from an analogue hardware implementation complexity point of view, the circuit contains \( N+2 \) summers, \( N \) switches, one integrator, one multiplier, one absolute function block and three sources of constant signals (or two sources of constant signals if \( x_0 = 0 \)). Note that the block \( \text{Abs} \) can be realized, for instance, by using a switch and an inverter. Therefore, the circuit presented can be implemented in modern hardware using such traditional electronic circuit components as analogue summers, multiplier, inverter, switches, integrator, and sources of constant voltage or current. For comparison, the previous continuous-time model of analogue KWTA neural circuit (A circuit) presented in [23] needs \( N+2 \) summers, \( N+2 \) switches, one integrator and four sources of constant signals (or three sources of constant signals if \( x_0 = 0 \)). An implementation of one of the simplest KWTA networks with a single state variable and the Heaviside step activation function (SSV & HSAF network), presented in [20], requires \( N+1 \) summers, \( N \) switches, one integrator and one source of constant signals. Thus, the hardware implementation complexity of the circuit described by a state equation (4) and by an output equation (5) is close to that of these comparable analogs.
We show herein below that in the case of distinct inputs, the proposed circuit can always identify them in accordance with the KWTA property (2). A resolution of the circuit is theoretically infinite and does not depend on its parameter values. Since the circuit is capable of correctly processing any finite value distinct inputs, its resolution is the same as that in other comparable neural networks with the same property [1], [20], [21], [23].

Since the present circuit can operate correctly with any finite initial condition \(-\infty < x_0 < \infty\), it requires neither a periodical resetting for repetitive processing of input sets, nor corresponding analogue supervisory circuit, nor spending additional processing time. This simplifies the hardware and decreases the convergence time to the KWTA operation.

IV. EXISTENCE AND UNIQUENESS OF THE KWTA STEADY-STATES

In the steady state mode, if \(dx/dt = 0\), the state equation (4) is transformed to the equilibrium equation given by

\[
\|x\| + p \left( \sum_{k=1}^{N} S_k(x) - K \right) = 0. \tag{10}
\]

Let us analyze the existence and uniqueness of the KWTA steady-states of the state equation (4) solutions on the basis of equilibrium equation (10). For this purpose, we formulate and prove the following theorem.

Theorem 1. Consider inequalities (1), the state equation (4), and equilibrium equation (10). If \(p > 0\), then there exist KWTA steady-states of the state equation (4) solutions.

Proof. According to (1), the elements of inputs \(a_n\), \(k = 1, 2, \ldots, N\) are distinct. Therefore, there exists such a constant number \(x^* \in \mathbb{R}\), that \(a_{K+1} \leq x^* < a_K\). For \(x^*\), equality \(\sum_{k=1}^{N} S_k(x^*) - K = 0\) holds meaning that \(x^*\) is a solution of equilibrium equation (10). Since inequality \(\|x\| + p > 0\) is satisfied for each \(p > 0\), equation (10) has the only solution \(x^*\). Taking into account (2) and (3), we can claim that there exist steady-state KWTA outputs \(b_{n_k} = a_{n_k} - x^*\), \(k = 1, 2, 3, \ldots, N\) of the model (4), (5). This means that \(x^*\) is a KWTA steady-state of the state equation (4) solution.

Corollary. The steady-state solution of the state equation (4) can take on any finite value in the range \(a_{K+1} \leq x^* < a_K\) satisfying the equality \(\sum_{k=1}^{N} S_k(x^*) - K = 0\). The solution \(x^*\) is not unique since this equality holds for each \(x^* \in [a_{K+1}, a_K]\). It follows that the outputs \(b_{n_k} = a_{n_k} - x^*\), \(k = 1, 2, 3, \ldots, N\) are not unique either. However, the KWTA property (2) of the model (4), (5) is determined by signs of outputs \(b_{n_k}\) rather than by its values. These signs are unique for any \(a_{K+1} \leq x^* < a_K\). Therefore, the model (4), (5) possesses a unique KWTA property (2).

V. CONVERGENCE ANALYSIS OF THE STATE-VARIABLE TRAJECTORIES

We perform a convergence analysis of the trajectories of the state equation (4) solutions to the KWTA operation by using the Lyapunov direct method [26], [27]. The following lemma states the availability of sufficient conditions that ensure global stability of these trajectories and its global convergence to the KWTA operation.

Lemma 1. Consider the state equation (4). If \(\alpha > 0\) and \(p > 0\), then for any initial value \(-\infty < x_0 < \infty\), the trajectories of the state equation (4) solutions are globally stable in the sense of Lyapunov and globally converge to the KWTA operation.

Proof. By a shift of the origin to the equilibrium point \(x = x^*\), the state equation (4) can be written in the following equivalent Persidskii type form:

\[
dr/dt = \alpha [r] + p \left( \sum_{k=1}^{N} S_k(r) - K \right), \tag{11}
\]

where \(r = x - x^*\). Let us consider the following Lyapunov function associated with (11):

\[
V(r) = \frac{\sum_{k=1}^{N} S_k(\lambda) - K}{\lambda}, \tag{12}
\]

where

\[
S_k(\lambda) = \begin{cases} 1, & \text{if } a_{n_k} - \lambda - x^* > 0; \\ 0, & \text{otherwise}. \end{cases}
\]

Observe that part \(\sum_{k=1}^{N} S_k(\lambda)\) of (12) is a scalar non-differentiable discontinuous integer-valued stepping descending function with two-side constraints given by \(0 \leq \sum_{k=1}^{N} S_k(\lambda) \leq N\). Since \(I \leq K < N\), we obtain \(-K \leq \sum_{k=1}^{N} S_k(\lambda) - K \leq N - K\). The function (12) is continuous, convex, bounded from below and has got a minimum at \(r = 0\). In accordance with (12), the upper-right time dini-derivative of \(V(r)\) is given by

\[
D^+ V(r) = \sum_{k=1}^{N} S_k(r) - K \tag{14}
\]

Consequently, the time derivative \(dV(r)/dt\), along the solution \(r(t)\) of (11) is determined as follows:

\[
dV/dt = D^+ V(r) dr/dt =
\]
\[ -\alpha|p| + p \left( \sum_{k=1}^{N} S_k(r) - K \right)^2. \] (15)

As we can see from (15), if \( \alpha > 0 \) and \( p > 0 \), then \( dV / dt < 0 \) if \( \sum_{k=1}^{N} S_k(r) - K \neq 0 \) and \( dV / dt = 0 \) if \( \sum_{k=1}^{N} S_k(r) - K = 0 \). Note that \( dV / dt = 0 \) implies that \( dr / dt = 0 \). In other words, if in some open interval of \( t \), the equality \( r(t) = 0 \) holds, then \( dr / dt \equiv 0 \) in that interval. For equation (11) at each \( dr / dt \neq 0 \), a derivative (15) is strictly negative definite, i.e., \( dV / dt < 0 \) under the conditions \( \alpha > 0 \), \( p > 0 \) except at the equilibrium points where it vanishes. This means that if \( \alpha > 0 \) and \( p > 0 \), then (12) is a monotonously decreasing function along each non-constant continuous-time trajectory \( r(t) \). Function \( V(r) \) is radially unbounded, i.e., \( V \rightarrow +\infty \) as \( |p| \rightarrow +\infty \). Therefore, according to the Lyapunov stability theory, it converges to a minimum and its derivative converges to zero \( dV / dt = 0 \). This proves that the point \( r = 0 \) (or equivalently \( x = x^* \)) is globally stable which means that the origin is globally attractive and the state dynamics of the model is globally stable. At the equilibrium point \( x = x^* \), the equalities \( \sum_{k=1}^{N} S_k(x^* + x^*) - K = \sum_{k=1}^{N} S_k(x^*) - K = 0 \) hold and inequalities (2) are satisfied demonstrating a KWTA property of the model (4), (5). Since the inputs are constrained in the range \(-\infty < a_{n_k} < \infty \), \( k = 1,2,\ldots,N \), the equilibrium point is bounded also in the range \(-\infty < x^* < \infty \).

Thus, for a monotonously descending Lyapunov function (12) bounded from below, i.e., for \( dV(r) / dt \leq 0 \) with \( dV / dt = 0 \) for \( dr / dt = 0 \), where the solution of the state equation (4) is changed, it follows that the inequalities \( \alpha > 0 \) and \( p > 0 \) are sufficient conditions for the trajectory of \( x(t) \) to be globally Lyapunov stable and globally convergent to the KWTA operation. Since the Peridiski type differential equation (11) is equivalent to the state equation (4), this result also holds for the trajectories of the original state equation (4) solutions.

It can be stated on the basis of the above derivation that there exists a time moment \( t^* > 0 \) such that for each \( t > t^* \), the following inequalities are satisfied:

\[ \infty > b_{n_1}(t) > \ldots > b_{n_k}(t) > 0 > b_{n_{k+1}}(t) > \ldots > b_{n_N}(t) > -\infty, \] (16)

where \( b_{n_k}(t) = a_{n_k} - x(t), \ k = 1,2,3,\ldots,N \). This means that the variable \( x \) of the state equation (4) starts from an initial value \( x_0 \) and finishes in the state \( x^* \) which corresponds to the components of the output vector \( b \) split into positive and negative planes in accordance with inequalities (2). At any time following \( t^* \), the outputs (3) possess the KWTA property.

VI. CONVERGENCE TIME TO THE KWTA OPERATION

Integrating both sides of the state equation (4) yields

\[ x(t) - x_0 = \alpha \int_{0}^{t} \left[ |x(t)| + p \left( \sum_{k=1}^{N} S_k(x(t)) - K \right) \right] dt. \] (17)

Since \( \sum_{k=1}^{N} S_k(x(t)) - K \) is an integer-valued stepping descending function if \( x(t) \) is not an equilibrium [20], then

\[ \alpha \left[ |x(t)| + p \left( \sum_{k=1}^{N} S_k(x(t)) - K \right) \right] \begin{cases} \geq +\alpha pt, & \text{if } x_0 < x^*; \\ \leq -\alpha pt, & \text{if } x_0 > x^*. \end{cases} \]

Therefore

\[ x(t) - x_0 \begin{cases} \geq +\alpha pt, & \text{if } x_0 < x^*; \\ \leq -\alpha pt, & \text{if } x_0 > x^*. \end{cases} \] (18)

The following upper bound of convergence time of the model state variable trajectories to the KWTA operation can be obtained from inequalities (18):

\[ t^* \leq \frac{x^* - x_0}{\alpha p}. \] (19)

As we can see from inequality (19), the convergence time is finite. Moreover, the upper bound (19) is inversely proportional to \( \alpha \) and \( p \). It means that the convergence speed of the model state variable trajectories to the KWTA operation increases with increasing values of parameters \( \alpha \) and \( p \).

Note that the model described by the state equation (4) and output equation (5) can also be used in the case of time-varying inputs \( a_{n_k}(t), \ k = 1,2,3,\ldots,N \) if the module of speed change of such inputs is much less than that of the state variable \( x \) trajectories in transient modes. In this case, condition

\[ \left| da_{n_k} / dt \right| < \left| dx / dt \right|. \] (20)

\( k = 1,2,\ldots,N \) should be satisfied for each \( t < t^* \). Note that since according to (4) \( \left| dx / dt \right| = \alpha \left[ |x| + p \left( \sum_{k=1}^{N} S_k(x) - K \right) \right] \frac{N}{2} \), therefore \( \left| dx / dt \right|_{\min} = \alpha p \) for any \( \alpha > 0 \), \( p > 0 \). For a worst case condition (20) is given by
\[ \left| \frac{da_n}{dt} \right|_{\text{max}} \ll \alpha p. \] (21)

Let us consider the example with corresponding computer simulations which illustrates the performance of the herein presented analogue KWTA neural circuit. We also compare a convergence time of the circuit state variable trajectories to the KWTA operation with that of some other competitive networks.

Example 1. We set 200 uniformly distributed random initial values \( x_0 \in [-250,250] \) of inputs \( a_{n_k}, k=1,2,3,\ldots,N \) uniformly distributed within the interval \([-250,250]\) for \( N=400, K=100, \alpha = 10^6 \), and \( p=1 \). A 1.81 GHz desktop PC and the variable order Adams-Bashforth-Moulton solver of non-stiff differential equations ODE113 with relative and absolute error tolerances equal to \( 10^{-5} \) are employed. Fig. 2 presents, in normalized units, the state variable transient behaviors showing that the state variable trajectories are globally stable and globally convergent to the KWTA operation from each initial value. For given data the maximal convergence time of the state variable trajectories to the KWTA operation of the proposed KWTA neural circuit compared with several others are listed in summary Table 1, where LP-based network is the linear programming problem based network, ID-network is the improved dual network. This table shows that the circuit proposed herein has higher convergence speed to the KWTA operation then that of these comparable analogue neural networks. Note that according to simulations this speed increases with increasing a number of inputs \( N \).

VII. SOME APPLICATIONS OF THE CIRCUIT

A. Internet Information Retrieval

In the internet information retrieval, the weights of \( N \) pages can be calculated using PageRank algorithm. Then, the PageRank results are displayed using, for instance, Quicksort algorithm which has \( O(N \log N) \) computational complexity on average. However, in the internet information retrieval, instead of sorting all the results, usually only the ten or twenty most important ones with largest weights should be found within numerous results [20], [29]. Such a problem can be formulated as choosing the \( K \) most important within \( N \) results, where \( 1 \leq K < N \). Let us employ the presented KWTA neural circuit which has \( O(N) \) computational complexity to solve this problem.

Example 2. We assume that after the search in the web, the vector of page weights was computed and the values of its elements can then be used to rank the search results. Let it be necessary to find \( K=20 \) pages with the largest weights within \( 1000 \) pages. We use the vector of weights from PageRank algorithm \( r_{n_k}, k=1,2,3,\ldots,1000 \) uniformly distributed within the interval \([0,16000]\) as an input vector of the KWTA neural circuit and the model (4), (5) with \( \alpha = 10^6 \), \( p=1 \), \( x_0=0 \). The simulation results show that the steady state outputs \( b_{n_k} = r_{n_k} - x_k, k=1,2,3,\ldots,1000 \) indicating the pages with the highest PageRank weights are achieved after the convergence time \( t^* < 0.002 \) s.

B. Parallel Sorting

As known, sorting is a fundamental operation of data processing [30]. In the case of parallel sorting, a sorting order can be represented as a permutation matrix. In such a matrix, “\( i \)” in the row labeled with \( a_i \) and column marked with \( c_j \) can be defined as the \( i \)-th item in an unsorted list and \( j \)-th item in a sorted list [20], [31]. In the case \( i=1,2,3,4,5 \), the corresponding permutation matrix

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

represents an unsorted list \( \{a_1, a_2, a_3, a_4, a_5\} \) and its ordered list \( \{a_2, a_3, a_5, a_4, a_1\} \). The above permutation matrix can be transformed to the sorting matrix
\[
S^1 S^2 S^3 S^4 S^5 \text{ rank}
\]
\[
a_1 = 0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 3
\]
\[
a_2 = 1 \quad 1 \quad 1 \quad 1 \quad 1
\]
\[
a_3 = 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 4
\]
\[
a_4 = 0 \quad 1 \quad 1 \quad 1 \quad 1 \quad 2
\]
\[
a_5 = 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 5
\]

It is not hard to see that in general case the sorting results can be presented by

\[
c_i = S_i, \quad c_{k+1} = S^{k+1} - S^k, \quad k = 1, \ldots, N-1. \tag{22}
\]

Let us determine the elements of \(j\)-th column

\[
S^j = [S_j^1, S_j^2, \ldots, S_j^N], \quad j=1,2,3,\ldots,N
\]

of the sorting matrix by step functions (7) using the state equation (4) of the KWTA neural circuit. If we use \(N-1\) equations (4) and each equation computes one column of the sorting matrix from left to right with \(K\) increasing from 1 to \(N-1\), then only \(N-1\) neurons will be necessary with a substantial reduction of the circuit complexity compared with the other analogue sorting networks with \(N^2\) neurons [6]. Specifically, the JWTA neural circuit can be used to determine the largest element of the list. Next, the 2WTA circuit computes the second item in the list without recounting the first item. As such, the whole list of \(N\) items can be sorted by using the \((N-1)\)WTA circuits without the need to compute the last item [20], [31].

**Example 3.** Let it be necessary to sort \(N=10\) inputs \(a=[7,1,5,9,4,6,2,8,10,3]^T\) using the neural circuit model (4). In this case, using nine models (4), only nine neurons are necessary in contrast to 100 neurons in the analogue sorting network in [6]. Fig. 3 shows the convergent behaviors of state variables in the nine models (4) for parallel sorting, where \(\alpha = 10^6, x_0 = 0\).

As we can see from the simulation results, the correct sorting of the inputs is achieved after the convergence time less than \(1.0 \times 10^{-6}\) s.

**C. Rank-Order Filtering**

Rank-order filters (ROF) are known to be nonlinear filters that identify, in time or spatial domain, the signal of \(k\)-th rank among all elements of an \(n\)-dimensional signal vector. ROFs have many applications, specifically in image processing, signal processing, speech processing, noise removal, computerized tomography, biomedical imaging, pattern recognition, coding and digital TV, etc. Numerous methods have been proposed to design ROFs [20], [32] – [35]. In particular, an analogue VLSI ROF built on two multiple-winners-take-all blocks that compute in parallel was proposed in [35]. A KWTA model with \(K\) winners is used in parallel to another KWTA model with \(K-1\) winners to select the input with its rank-order being \(K\) in [20].

We present the output signal of ROF as

\[
c = a^T (S_k^{K-1} - S_k^{K-1}), \tag{23}
\]

where \(S_k^k, k=1,2,\ldots,N\) is the step function (7) determined for \(K\) winners, \(S_k^{K-1}(x), k=1,2,\ldots,N\) is the step function (7) obtained for \(K-1\) winners, \(c=[c_n, c_{n+1}, \ldots, c_{n+6}]^T\) is an output vector of the ROF.

**Example 4.** In order to demonstrate the performance of the ROF described using the expression (23) and the state equation (4), we consider the simulation given below. The input signals are set as \(a_i = A \sin(i \omega + \phi) + d, \quad i=1,2,\ldots,N\)

where \(\omega\) is the angular frequency, \(\phi\) is the phase shift, and \(d\) is the bias. We also use random noise \(n\) uniformly distributed on the interval [\(-A, A\)] as an additional input signal \(n=a_n\). In order to improve the solution of the state equation (4), we employ a corresponding finite difference equation with the time step \(\Delta t = 2.0 \times 10^{-7}\). The first subplot in Fig. 4 presents eight input sinusoidal signals, the second subplot shows the noise \(n\), and the last subplot depicts the filtered output signal \(c\) obtained using the finite difference equation corresponding to the state equation (4) and the expression (23) for \(N=9, K=5, \alpha = 10^6, p=1, x_0 = 0, A=5, d=10, \omega = 2000\pi, \phi = \pi/4\). As we can see from this figure, the filtered output signal \(c\) presents a phenomenon of chattering in the time points where the condition (1) is violated.

![Fig. 3. Transient states of five models (4) for parallel sorting with inputs](https://example.com/fig3)

In order to improve the ROF performance, let us consider the case when the two or more inputs are equal to one another. If such inputs belong to \(K\) winners or to \(N-K\) losers, then, as it was shown above, the outputs of the model converge to the KWTA operation if \(\alpha > 0, \quad p > 0\). However, in case the model should distinguish equal maximal inputs and split them into positive and negative planes if \(K\) maximal inputs do not exist, then the outputs of the model which do not have the KWTA property will be obtained. In particular, as we can see from the simulation results shown in Fig. 4, the model outputs can oscillate in the time points in which the inequalities (1) are violated, in other words, where input signals are equal to one another.
In order to remove the mentioned oscillations, let us generalize the expression (23) to the case of processing such time-varying input signals which can be equal to one another in some time points. Since in these time points inequalities (1) are not satisfied, $K$ maximal input signals do not exist. In order to improve the ROF performance, we extend the expression (23) to the following form:

$$c = a^T \left( S^K - S^{K-1} \right), \text{if } R^M(x) = 0, R^{M-1}(x) = 0;$$
$$dc/dt = 0 , \ c_0 = 0 , \text{otherwise},$$

where $R^M(x)$ is a residual function (6) for $K=M$, $c_0 = 0$ is an initial condition. It is not hard to see that in the KWTA steady state mode, the system of algebra-differential equations (24) is reduced to the expression (23) which is a particular case of the system (24). In the transient mode, output signals of the ROF should be described by the degenerative differential equation of the system (24).

Example 5. For the same data as in the Example 4, the filtered output signal $c$ obtained using the system of algebra-differential equations (24) and the finite difference equation corresponding to the state equation (4) is shown in Fig. 5. As we can see from this figure, the oscillations have been completely removed in the time points where input signals are equal to one another. Note that in this example inequality (21) is satisfied in the transient modes of the state variable $x$ solution trajectories since $\left| dx/dt \right|_{\text{max}} = 10000 \pi$ and $\left| dx/dt \right|_{\text{min}} = 10^6$. According to simulation results presented in Fig. 5, the analogue ROF built on the basis of the system of algebra-differential equations (24) and the state equation (4) demonstrates a good performance including the time points of equal input signals.

VIII. CONCLUSION

This paper presents a continuous-time mathematical model and a corresponding functional block-diagram of an analogue $K$-winners-take-all neural circuit. In contrast to the predecessor, the proposed KWTA neural circuit is capable of selecting $K$ maximal among any unknown finite value $N$ distinct inputs located in an unknown range, where $I \leq K < N$. Computer simulations show that the convergence time of the circuit state variable trajectories to the KWTA operation is less than that in the previous KWTA circuit model and in one of the simplest KWTA models. The hardware implementation complexity of the proposed KWTA circuit is close to that in these comparable analogs.

Fig. 5. The dynamics of eight sinusoidal input signals $a_i$, $i=1,2,...,N-1$, one noised input signal $n$, and filtered output signal $c_K$ of the ROF described using the system of algebra-differential equations (24) and the finite difference equation corresponding to the state equation (4) – Example 5.

Since the circuit presented has a high convergence speed to the KWTA operation, it can be used for reducing the data processing time, for accelerating the digital image and speech processing, for coding and for digital TV [9], [20]. Further investigations are directed towards the circuit implementation using an up-to-date hardware and its various applications.

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REFERENCES
